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We define and explore the classical counterpart of entanglement in complete analogy with quantum mechanics. Using a basis independent measure of entropy in the classical Hilbert space of densities that are propagated by the Frobenius-Perron operator, we demonstrate that at short times the quantum and classical entropies share identical power laws and qualitative behaviors.

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Quantum entanglement has been studied with renewed vigor in the recent past due to its relevance in quantum information theory [1]. It has been recognized as an unique quantum resource that could be utilized for potentially powerful and qualitatively different quantum computers [2]. It measures the degree of non-separability between individual parts of a composite system. That this non-separability may have apparently dramatic effects was implicitly exploited in the Einstein, Podolsky, Rosen paradox. It appears that much of the strangeness of quantum phenomena follows from the existence of such entanglement. A classical system, composed perhaps of very many interacting parts, could have at all times a unique state assigned to these parts, in terms of their phase-space variables. However it is quite clear that a quantum state is not the counterpart of a classical phase-space point; although both describe the maximum possible information we can have of the system within the framework of the respective theories.

To arrive at the classical counterpart of entanglement we study a classical measure of non-separability of individual subsystems. In the spirit of entanglement we do not allow for transformations that may mix the states of the individual subsystems, i.e., the allowed transformations must be local. A Hilbert space formulation of classical mechanics is the natural starting point towards this end. Recently, classical mechanics on the Hilbert space has been studied to understand certain aspects of quantum-classical correspondence, especially in the presence of classical chaos [3]. The principal object is the Frobenius-Perron (F-P) operator, that is the exponentiation of the Liouville operator, which time evolves phase-space densities and is the classical counterpart of the quantum unitary propagator. States in the Hilbert space are allowed phase-space densities, and the measure of separability is simply the separability of evolving phase-space densities. Thus the machinery of quantum entanglement, measuring entanglement in a complex Hilbert space, may be applied directly within this space of classical densities. Recent work, with the aim of bringing into sharper focus the unique nature of quantum entanglement, has explored the classical part of correlations from an information theoretic point of view [4].

Another related aspect is the issue of entropy production, and the definition of entropy in irreversible processes [5]. The usual definition of entropy in an ensemble through the integral version of the Shannon entropy, runs into conceptual difficulties. As is well known it is strictly a constant and thus some sort of coarse-graining is needed to define a changing entropy. Entropies are then defined through partially integrated out densities, say through the Bogoliubov hierarchy [6]. On the other hand a von Neumann entropy of the classical densities, as we define below, will be non-negative by construction, need not be a constant of motion, and may be thought of as the most natural possible coarse-graining. A related work in this context which explores quantum-classical correspondence for open chaotic systems is in [7].

Due to the difficulty in defining a measure of entanglement for multipartite spaces, we consider a system having just two particles. Quantum mechanically the system is described by states in a bipartite Hilbert space and as is well known there exists a Schmidt decomposition which if it has more than one non-zero term implies entanglement, and the converse is also true [1]. Now consider the classical mechanics as described in the phase-space  $(q_i, p_i)$ ,  $i = 1, 2$ . The flow is generated by a Hamiltonian  $H$  and the F-P operator  $\mathcal{P}$  is defined through:

$$\mathcal{P}(t) = \exp(-t\{ \cdot, H \}); \rho(q_i, p_i, t) = \mathcal{P}(t) \rho(q_i, p_i, 0). \quad (1)$$

The F-P operator is unitary. Rather than considering a Hilbert space of  $L^1$  functions, for simplicity we consider the space of  $L^2$  functions. For instance we may define the evolution of the square-root of a classical density. This also seems natural in view to our wanting to do comparisons with quantum entanglement. Let  $\{F_{m,n}(q, p)\}$  be an orthonormal basis set in such a Hilbert space, which we will write compactly as  $\{|m, n\rangle\}$ . We consider the case where there is a denumerably infinite number of such basis functions labeled by the pair  $(m, n)$ . A pair is used to facilitate defining a basis in each Hilbert space corresponding to each phase-space variable. Thus we can consider the classical Hilbert space as being an outer product of four Hilbert spaces. However in the following we will consider it as a product space of two Hilbert spaces, each corresponding to one degree of freedom.

Expanding any density  $\rho$  in the four-dimensional phase space in terms of these functions we may write:

$$|\rho\rangle = \sum a(m_1, n_1; m_2, n_2) |m_1, n_1\rangle |m_2, n_2\rangle, \quad (2)$$

where the  $a(\cdot)$  are expansion coefficients. Now the “reduced density matrix” is simply the partial trace:

$$\hat{\rho}_1 = \text{Tr}_2(|\rho\rangle\langle\rho|), \quad (3)$$

while the von Neumann entropy of this reduced density matrix is the entropy of classical entanglement and is given by:

$$S_C = \text{Tr}_1(\hat{\rho}_1 \log(\hat{\rho}_1)). \quad (4)$$

The analogy to the quantum definitions are evident and the existence of a Schmidt decomposition follows. The entropy  $S_C$  can be zero or positive, if it is zero there is no entanglement, and the densities are separable into each degree of freedom, while if it is positive this is not possible. The basis independence of the entropy  $S_C$  follows from the two trace operations, and thus it represents a physically meaningful quantity, which by its construction ought to qualify as the classical counterpart of quantum entanglement.

We now provide evidence for this by means of an example. We choose two coupled standard maps for the possibility of studying potentially interesting connections between deterministic chaos and entanglement [8]. In fact the motivation for the present work is derived from a previous work [9] where we studied a similar system which showed that classical chaos enhances quantum entanglement. Furthermore we showed that some Schmidt vectors were “scarred” by projections of certain classical periodic orbits. A study of coupled kicked tops has also shown that quantum entanglement increases linearly in time at a rate which is the sum of the classical positive Lyapunov exponents [10]. These are indications that quantum entanglement is not entirely immune to the lure of the classical.

Consider the classical map defined on the four-torus  $T^2 \times T^2$ :

$$\begin{aligned} q'_i &= q_i + p_i \pmod{1} \\ p'_i &= p_i - \partial V / \partial q'_i \pmod{1}, \end{aligned} \quad (5)$$

where  $i = 1, 2$  and the potential  $V$  is

$$\begin{aligned} V = & -\frac{K_1}{(2\pi)^2} \cos(2\pi q_1) - \frac{K_2}{(2\pi)^2} \cos(2\pi q_2) + \\ & \frac{b}{(2\pi)^2} \cos(2\pi q_1) \cos(2\pi q_2) \end{aligned} \quad (6)$$

This is a symplectic transformation on the torus  $T^4$  and may be derived from a kicked Hamiltonian in the standard manner [8]. This is somewhat different from the previously studied four-dimensional generalizations of the

standard map, and has been chosen to maximize symmetries. The parameter  $b$  controls the interaction between the two standard maps, while  $K_i$  determine the degree of chaos in the uncoupled limit. The advantage in studying such maps is our ability to write explicitly the relevant unitary operators, both classical and quantal.

The classical unitary F-P operator over one iteration of the map is

$$\mathcal{P} = \mathcal{P}_1 \otimes \mathcal{P}_2 \mathcal{P}_b, \quad (7)$$

where  $\mathcal{P}_i$  is the F-P operator for the standard map on  $T^2$  and acts on the single-particle Hilbert spaces, while  $\mathcal{P}_b$  is the interaction operator on the entire space. We use as a single-particle basis the Fourier decomposition:

$$F_{m,n}(q,p) = \exp(2\pi i(mq + np)). \quad (8)$$

Then the matrix elements of the F-P operators are:

$$\langle m'_i, n'_i | \mathcal{P}_i | m_i, n_i \rangle = J_{m'_i - m_i}(K_i n'_i) \delta_{n_i - m_i, n'_i}, \quad (9)$$

and

$$\begin{aligned} \langle m'_1, n'_1; m'_2, n'_2 | \mathcal{P}_b | m_1, n_1; m_2, n_2 \rangle = \\ J_{l+} \left( \frac{b(n_1 + n_2)}{2} \right) J_{l-} \left( \frac{b(n_1 - n_2)}{2} \right) \delta_{n_1, n'_1} \delta_{n_2, n'_2}, \end{aligned} \quad (10)$$

where

$$l_{\pm} = \frac{1}{2} ((m_1 - m'_1) \pm (m_2 - m'_2)),$$

and the delta functions are Kronecker deltas. The orders of the Bessel J functions are restricted to the integers.

If the initial density is described by a vector (Eq. (2))  $|\rho(0)\rangle$  then

$$|\rho(T)\rangle = \mathcal{P}^T |\rho(0)\rangle \quad (11)$$

is the density after a time  $T$ . The same may of course be equivalently obtained by means of the Liouville equation. The number of classical components excited increases with time in general including increasingly larger frequencies. This is indicative of the fine structure that is being created in phase-space by the stretching and folding mechanism. The frequencies involved appears to increase exponentially in time for chaotic systems, while we may expect a polynomial growth for integrable or near-integrable systems. The Arnold cat map provides a completely chaotic exactly solvable model where this exponential increase is readily seen [11], while the free rotor, corresponding to the case  $K_i = 0$  for the standard map on  $T^2$  is an exactly solvable integrable model where a linear growth is apparent. With the flow of probability moving inexorably outward, for chaotic systems, the infinite dimensional Hilbert space of classical mechanics provides a setting for an approach to equilibrium, with

every smooth density tending towards the invariant density in a weak sense [12].

The quantization of the symplectic transformation in Eq. (5) is a finite unitary matrix on a product Hilbert space of dimensionality  $N^2$ , and  $N = 1/h$ , where  $h$  is a scaled Planck constant. The classical limit is the large  $N$  limit. The quantization is straightforward as there exists a kicked Hamiltonian generating the classical map [8]. The quantum standard map on the two-torus in the position representation is

$$\langle n'_i | U_i | n_i \rangle = \frac{\exp(-i\pi/4)}{\sqrt{N}} \exp \left( iN \frac{K_i}{2\pi} \cos \left( \frac{2\pi}{N} (n_i + \frac{1}{2}) \right) \right) \times \exp \left( \frac{\pi i}{N} (n_i - n'_i)^2 \right). \quad (12)$$

The position kets are labeled by  $n = 0, N-1$  and the position eigenvalues are  $(n + 1/2)/N$  while the momentum eigenvalues are  $m/N$ ,  $m = 0, \dots, N-1$ . The quantum phases have been chosen to maximize symmetries, and  $N$  is taken to be even. The four-dimensional quantum map is but a simple extension:

$$U = U_1 \otimes U_2 U_b, \quad (13)$$

which is the quantum equivalent of the classical F-P operator in Eq. (7) and the interaction operator is

$$\langle n'_1, n'_2 | U_b | n_1, n_2 \rangle = \exp \left[ -iN \frac{b}{2\pi} \cos \left( \frac{2\pi}{N} (n_1 + \frac{1}{2}) \right) \times \cos \left( \frac{2\pi}{N} (n_2 + \frac{1}{2}) \right) \right] \delta_{n_1, n'_1} \delta_{n_2, n'_2}. \quad (14)$$

$U$  is a unitary matrix and will induce mixing between the two subsystems. Some properties of such unitary matrices especially in connection with quantum entanglement has been studied previously [9].

For the initial state a product state composed of coherent states is a natural choice, as the corresponding initial classical density becomes apparent. We choose

$$|\psi(0)\rangle = |00\rangle \otimes |00\rangle, \quad (15)$$

where  $|00\rangle$  is a coherent state that is peaked at the classical point ( $q = 0, p = 0$ ) [13], which also happens to be a classical fixed point for the dynamics. The initial classical density will be a circular (periodicised) gaussian of width  $\sigma$  around the origin. We choose

$$a_0(m, n) = (8\pi\sigma^2)^{1/2} \exp(-4\pi^2\sigma^2(m^2 + n^2)), \quad (16)$$

to be the Fourier components of such a (approximate) periodicised gaussian. The initial density's components are then specified by the outer product

$$a_0(m_1, n_1; m_2, n_2) = a_0(m_1, n_1) a_0(m_2, n_2). \quad (17)$$

For  $\sigma \sim 1/\sqrt{N}$  the Husimi distribution of the quantum state is similar to the (square of the) classical initial density. For quantum-classical correspondence we require large  $N$  or small  $\sigma$ .

We time evolve the initial states, both quantum and classical, using the respective propagators and calculate the entanglement in each. Due to the infinite dimensionality of the classical Hilbert space and the at least polynomially increasing frequency components, the classical calculations in a truncated Hilbert space lead to rapid loss of accuracy. Thus this work, whose main intention is to compare the classical entanglement as defined in Eq. (4) with quantum entanglement, merely touches the surface of interesting results. We put  $K_1 = K_2 = 0$  and turn on the interaction parameter  $b$  only slightly. Thus the uncoupled systems are free rotators, while the coupled system is near-integrable. For sufficiently large values of the interaction, full fledged chaos seems to develop, a case which we do not discuss further here.

First we look at entanglement produced for very short times, as this is numerically easily accessible. We study this entanglement as a function of the interaction. In Fig. 1 is shown entanglement produced at times 1 and 2.

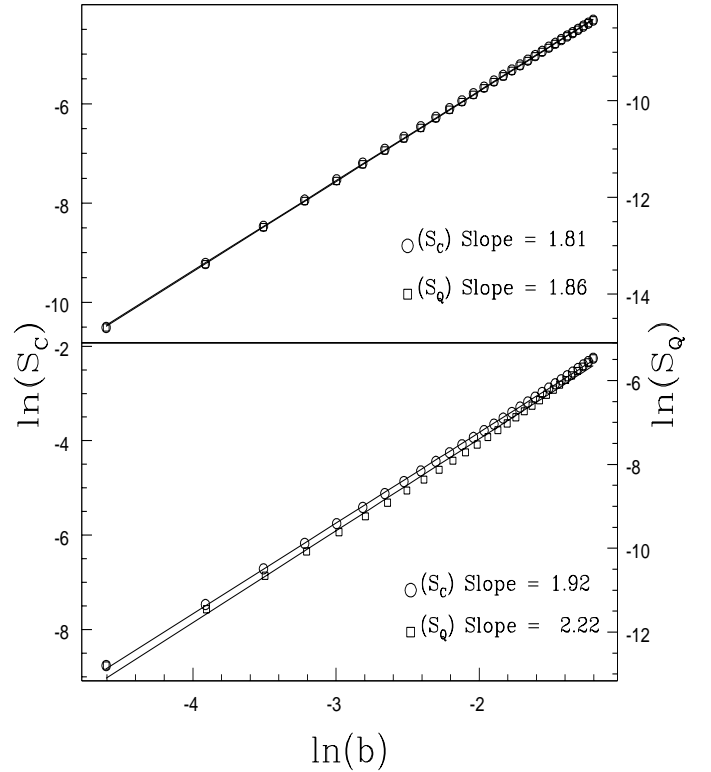


FIG. 1. Classical and quantum entanglement after time  $T = 1$  (top) and  $T = 2$  (bottom) as a function of the interaction for the initial state described in the text. The quantum calculations correspond to  $N = 50$ , while the classical initial state has  $\sigma = 0.1$ .

We find approximate power laws that are nearly the same for both classical and quantum entanglement. The best correspondence is naturally at the shortest time,  $T = 1$ , when a power law of the form  $S \sim b^{1.8}$  holds in both cases, while at time 2 the exponents are not so close. However at this time the power law seems to be approximate and the deviations from the straight line is seen to have common features in both the classical and quantum cases. The  $N$  or  $\hbar$  dependence of the quantum entanglement does not seem to be a power law at these short times and we do not pursue this here, thus we do not compare the absolute values of the two entanglement.

In Fig. 2 we plot both the entanglements as a function of time for a fixed value of the interaction. We see qualitatively similar behaviour. The quantum entanglement tend to saturate at higher values for larger  $N$ , while the classical entanglement seems to be continuously increasing. This is however difficult to see numerically due to increase of errors, for instance at the highest time shown in the figure,  $T = 6$ , the normalization deteriorated from unity to about .9948. More elaborate work is needed to establish the properties of classical entanglement.

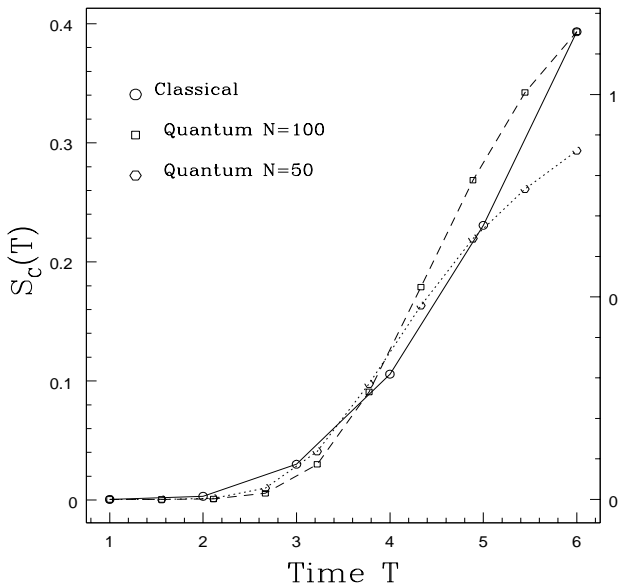


FIG. 2. Classical and quantum entanglement as a function of time for the same initial state as in Fig. 1. The interaction  $b = 0.05$ .

We focus finally on an interesting regime where both the subsystems are chaotic, while the interaction is weak. In this case a linear quantum entropy increase was found, with the rate being proportional to the sum of positive Lyapunov exponents [7,10]. In fact this linear regime is time-bound and gives way to saturation, while the entropy in this linear regime scales as  $N^2$  ( $\hbar^{-2}$ ). This is illustrated in Fig. 3 for the system we consider in this paper.

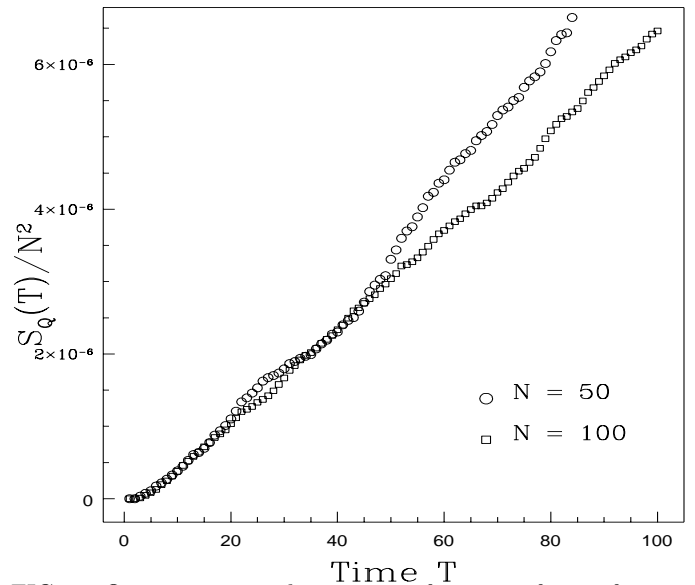


FIG. 3. Quantum entanglement as a function of time for the same initial state as in Fig. 1.  $K_1 = 6.0$ ,  $K_2 = 5.0$ , while The interaction is  $b = 0.001$ . The linear regime is seen to scale as  $N^2$ .

The linear entropy increase indicates an exponentially rapid increase in the number of single-particle basis states used. Thus we conjecture that this regime is in fact a classical one, with this being a reflection of the exponential increase in the number of basis states explored in the classical Hilbert space. The rate being proportional to the sum of positive classical Lyapunov exponents seems to suggest that the classical entanglement production rate is proportional to the classical K-S entropy. The subsequent saturation is a quantum effect, while we may expect the classical entropy to continue to linearly increase. We further conjecture that the difference between chaotic and near-integrable or integrable systems will be reflected in the classical entanglement  $S_C(T)$  behaving at large times as either  $T$  or  $\log(T)$ , respectively. Much larger classical calculations will be able to prove or disprove these statements.

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